Recall that a (undirected) graph $G$ is defined as an ordered pair $(V, E)$ where $V$ is a finite non-empty set, and $E \subseteq V^{(2)}$ is a set edges (two-element subsets of $V$). The elements of $V$ are the vertices of $G$, and the elements of $E$ are the edges of $G$. It is convention to refer to $|V|$ as $n$ and $|E|$ as $m$.

An undirected bipartite graph $G$ is defined as an ordered pair $(V, E)$ where $V$ can be partitioned into two sets $A$ and $B$ such that $E \subseteq \{ \{u, v\} \mid u \in A, v \in B \}$.

1. Prove by induction that, for any undirected graph $G = (V, E)$, the number of edges in $G$ is twice the sum of degrees of all vertices in $G$, i.e. $\sum_{v \in V} d(v) = 2m$.

**Solution:** We prove this by induction on the number of edges in the graph. That is, we will prove that for every non-negative integer $m$, the claim holds for any graph with $m$ edges.

Let $m$ be an arbitrary non-negative integer. If $m = 0$, then every vertex in any graph $G$ with zero edges has zero degree, so the claim holds in this case.

Otherwise, $m > 0$. Assume that the claim holds for all undirected graphs with less than $m$ edges. Let $e = \{a, b\}$ be some edge in $E$, and let $G' = (V, E \setminus \{e\})$ be the graph obtained by removing $e$ from $G$. Then $G'$ has $m - 1 < m$ edges, so $\sum_{v \in V} d'(v) = 2(m - 1)$ by the induction hypothesis where $d'(v)$ denotes the degree of $v \in V$ in graph $G'$. Note that $d'(u') = d(u)$ for every vertex $v \in V \setminus e$; that is, any vertex that is not an endpoint of $e$ has the same degree in $G$ and $G'$. Similarly, see that $d(a) = d'(a) + 1$ and $d(b) = d'(b) + 1$ since $a$ and $b$, the endpoints of edge $e$, have one more incident edge in $G$ than in $G'$, namely $e$. It follows that $\sum_{v \in V} d(v) = 2 + \sum_{v \in V} d'(v) = 2 + 2(m - 1) = 2m$ as desired. 

2. Prove by induction that, for any undirected bipartite graph $G = (V, E)$ with bipartition $A$ and $B$, $\sum_{v \in A} d(v) = \sum_{v \in B} d(v) = m$.

**Solution:** We will prove this by induction on the number of edges in the graph. That is, we will prove that for every non-negative integer $m$, the claim holds for any graph with $m$ edges.

Let $m$ be an arbitrary non-negative integer. If $m = 0$, then the degree of all vertices is also 0, so the claim holds in this case.

Otherwise, $m > 0$. Assume the claim holds for all undirected bipartite graphs with less than $m$ edges. Consider any undirected bipartite graph $G = (V, E)$ with bipartition $A$ and $B$ and $m$ edges. Let $e = \{a, b\}$ be an arbitrary edge in $G$ for some $a \in A$ and $b \in B$, and let $G'$ be the undirected graph obtained by removing edge $e$. Clearly $G'$ is bipartite with bipartition $A$ and $B$ and has $m - 1$ edges, so $\sum_{v \in A} d'(v) = \sum_{v \in B} d'(v) = 2(m - 1)$ by the induction hypothesis where $d'(v)$ denotes the degree of vertex $v$ in $G'$. Note that $d(v) = d'(v)$ for any vertex $v \notin \{a, b\}$, $d(a) = d'(a) + 1$, and $d(b) = d'(b) + 1$ since the only difference between $G$ and $G'$ is the edge $\{a, b\}$. It follows that $\sum_{v \in A} d(v) = 1 + \sum_{v \in A} d'(v)$ and $\sum_{v \in B} d(v) = 1 + \sum_{v \in B} d'(v)$ which immediately implies $\sum_{v \in A} d(v) = \sum_{v \in B} d(v)$ by the induction hypothesis. Finally, every vertex $v \in V$ is in exactly one of $A$ or $B$, so from problem 1 we have $\sum_{v \in A} d(v) + \sum_{v \in B} = 2m$. It follows that $\sum_{v \in A} d(v) = \sum_{v \in B} = m$ as desired.
3. Prove by induction that, for any binary string $s$ that begins with a 1 and ends with a 0, there is a 1 immediately before a 0 somewhere in $s$.

**Solution:** We prove this by induction on the length of the string. That is, we will prove that for every integer $n \geq 2$, the claim holds for any string of length $n$.

Let $n$ be an arbitrary positive integer. If $n = 2$, there is only one string of length 2 that begins with a 1 and ends with a 0, namely $s = 0.1$. Clearly the claim holds for $s$ in this case.

Otherwise $n > 2$. Assume the claim holds for all strings that begin with a 1, end with a 0, and have length $k$ such that $2 \leq k < n$. Let $s$ be an arbitrary string of length $n$ that begins with 1 and ends with 0. Since $n > 2$, $s = 1.a.t.0$ where $a \in \{0, 1\}$ and $t$ is a string of length $n - 3$. There are two cases for $a$:

(a) If $a = 0$, then $s = 1.0.t.0$ and the first 1 in $s$ is immediately before the first 0 in $s$, so we are done.

(b) If $a = 1$, then $s = 1.1.t.0$. Let $u = 1.t.0$ which has length $n - 1$, begins with a 1, and ends with a 0. Since $2 \leq n - 1 < n$, the induction hypothesis implies $u$ contains a 1 immediately before a 0, and thus $s = 1.u$ contains a 1 immediately before a 0.

4. A rooted binary tree is **full** if every node has either zero or two children. Prove that any rooted full binary tree with $i$ internal nodes (those with at least one child) has $2i + 1$ total nodes.

**Solution:** We prove this by induction on the number of internal nodes. That is, we will prove that for every integer $i \geq 0$, the claim holds for any rooted full binary tree with $i$ internal nodes.

Let $i$ be an arbitrary non-negative integer. If $i = 0$, then the only tree with no internal nodes is the tree with a single root node which has no children. $2(0) + 1 = 1$ so the claim holds in this case.

Otherwise, $i > 0$. Assume that the claim holds for all rooted full binary trees with less than $i$ internal nodes. Let $T$ be an arbitrary rooted full binary tree with $i$ internal nodes. Since $i > 0$, the root node of $T$ has children, specifically two since $T$ is full. Let the subtrees rooted at the two children be $T_1$ and $T_2$, and let $j$ be the number of internal nodes in $T_1$. Any non-root node of $T$ is internal in $T$ if and only the node is internal in $T_1$ or $T_2$. This implies the number of internal nodes in $T_2$ is $(i - 1) - j$. Since $T$ is full, $T_1$ and $T_2$ must be full. Since $0 \leq j < i$ and $0 \leq i - 1 - j < i$, the induction hypothesis implies $T_1$ has $2j + 1$ nodes and $T_2$ has $2(i - 1 - j) + 1 = 2i - 2j - 1$ nodes. It follows that $T$ has $(2j + 1) + (2i - 2j - 1) + 1 = 2i + 1$ nodes.