Recall that an undirected graph $G$ is defined as an ordered pair $(V, E)$ where $V$ is a finite non-empty set, and $E \subseteq V^{(2)}$ is a set edges (two-element subsets of $V$). A directed graph is defined similarly where the edge set $E$ is a subset of $V \times V \setminus \{(v, v) \mid v \in V\}$. The elements of $V$ are the vertices of $G$, and the elements of $E$ are the edges of $G$. It is convention to refer to $|V|$ as $n$ and $|E|$ as $m$.

A directed graph $G$ is defined as an ordered pair $(V, E)$ where $V$ is a finite non-empty set, and $E \subseteq V \times V$ is a set of length-two sequences of elements of $V$. In this class, we further restrict $E$ not to have elements of the form $(v, v)$ for any $v \in V$ (no self-loops). The elements of $V$ are the vertices of $G$, and the elements of $E$ are the edges of $G$. It is convention to refer to $|V|$ as $n$ and $|E|$ as $m$.

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1. Prove that an undirected graph $G = (V, E)$ is a tree if and only if there exists a unique path between every pair of distinct vertices in $G$.

**Solution:** First we prove that an undirected graph $G$ is acyclic if and only if there is *at most one* path between every pair of distinct vertices in $G$.

- $(\Leftarrow)$ We prove this implication by proving its contrapositive: if an undirected graph $G$ contains a cycle ($G$ is not acyclic), there is some pair of distinct vertices in $G$ such that there is more than one distinct path between them. To see this, let $G$ be any undirected graph $G$ that contains a cycle, and consider two vertices of a cycle in $G$. Clearly there are two distinct paths between these vertices as given by the edges of that cycle.

- $(\Rightarrow)$ We prove this implication by proving its contrapositive which we prove by smallest counterexample (WOP): for any undirected graph $G$, if there are at least two distinct paths between a pair of vertices in $G$, then $G$ contains a cycle. For sake of contradiction, let $G$ be an undirected graph with two distinct paths $P_1$ and $P_2$ between two distinct vertices $u$ and $v$ where the number of edges in $G$ is minimized. If the paths $P_1$ and $P_2$ only share vertices $u$ and $v$, i.e. they are interior-disjoint, then $P_1$ concatenated with $P_2$ is a cycle, a contradiction.

Henceforth, we assume $P_1$ and $P_2$ are not interior-disjoint, so they share a vertex $x$ that is neither $u$ nor $v$. Let $P_{1u}^v$ and $P_{1v}^u$ be the (sub)paths between $u$ and $x$ and between $x$ and $v$ contained in $P_1$, respectively, and let $P_{2u}^v$ and $P_{2v}^u$ be the (sub)paths between $u$ and $x$ and between $x$ and $v$ contained in $P_2$, respectively. Note that all of these subpaths must contain at least one edge. There are two cases to consider.

- If $P_{1u}^v$ and $P_{2u}^v$ are not equivalent, the subgraph $G'$ obtained by removing the edges of $P_{2u}^v$ and $P_{2v}^u$ from $G$ has less than $m$ edges and contains two distinct paths between $x$ and $v$, namely $P_{1v}^u$ and $P_{2v}^u$, which contradicts the choice of $G$.

- Otherwise, since $P_1$ and $P_2$ are not equivalent, it must be that $P_{1u}^v$ and $P_{2u}^v$ are not equivalent. Similar to the above, the subgraph $G'$ obtained by removing the edges of $P_{1u}^v = P_{1v}^u$ from $G$ has less than $m$ edges and contains two distinct paths between $x$ and $v$, namely $P_{1v}^u$ and $P_{2v}^u$. Again, this contradicts the choice of $G$.

In either case, a contradiction is reached, so no such $G$ exists and the claim holds.

With the fact above it is easy to finish the proof. From lecture, we know an undirected graph $G$ is connected if and only if there is *at least one* path between every pair of vertices.
Thus, it follows from the above that any undirected graph \( G \) is both connected and acyclic if and only if it has both \( \geq 1 \) path and \( \leq 1 \) path between every pair of vertices, which is to say there is exactly 1 (unique) path between every pair of vertices. Finally, since we also know an undirected graph is a tree if and only if it is connected and acyclic, we are done by a chain of equivalences.

\[ \blacksquare \]

2. Prove that for any finite partial order \( R \) on a finite set \( A \), there exists a chain decomposition of \( R \) on \( A \) of size at most the size of the longest antichain in \( R \). (This is one direction of Dilworth’s theorem; we previously argued the converse in lecture.)

**Solution:** Before we begin, we note that the following is a (less concise) proof adapted from one found online here. In recitation, we phrased the proof in terms of reachability in DAGs, specifically Hasse diagrams.

We will give a proof by induction on the size of the set \( A \). Let \( R \) be a finite strict partial order on a non-empty finite set \( A \) (the claim holds vacuously for empty \( A \)), and let \( n \) be the number of pairs in \( R \).

Consider the case where \( R \) is empty, i.e. \( n = 0 \). This case is trivial since all pairs of distinct elements are incomparable, and thus the largest antichain is \( A \) itself while there is chain decomposition of the same size where each element of \( A \) is placed in its own set. Thus, in this case, the claim holds.

Henceforth, we assume \( R \) is non-empty (and thus \( A \) is non-empty). Assume that the claim holds for all strict partial orders on any set with less than \( n \) pairs. Let \( s \in A \) be an element where \( \neg aRs \) for all \( a \in A \) and there exists \( b \in A \) such that \( sRb \); it can be easily proven such an \( s \) always exists. Let \( t \in A \) be an element where \( sRt \) and \( \neg tRa \) for all \( a \in A \); again, it can be easily proven such a \( t \) must exist. Clearly \( C = \{s, t\} \) is a chain since \( sRt \).

Now consider the set \( A' = A \setminus C \) and relation \( R' = R \cap (A' \times A') \); in other words, let \( R' \) be the relation obtained by removing all pairs containing \( s \) or \( t \) from \( R \). Note that \( R' \) is a strict partial order on set \( A' \) by the choices of \( s \) and \( t \). Let \( T' \) be an antichain of \( R' \) on \( A' \). Since \( |R'| < |R| \), there exists a chain decomposition \( CD' \) of \( R' \) on \( A' \) where \( |CD'| \leq |T'| \) by the induction hypothesis. This immediately implies a chain decomposition \( CD' \cup \{C\} \) of \( R \) on \( A \) since \( C \) is a chain and the only elements of \( A \) not in any chains of \( CD \) are exactly \( s \) and \( t \). Now let \( T \) be a longest antichain of \( R \). There are two cases.

- If \( |T'| < |T| \), we have \( |CD \cup \{C\}| = |T'| + 1 \leq |T| \), so we are done. Indeed, \( CD \cup \{C\} \) is a chain decomposition of \( R \) on \( A \) of size at most \( |T| \) where \( T \) is a longest antichain of \( R \) on \( A \).

- Otherwise, \( |T'| \geq |T| \). In this case, the argument of the previous case does not work; we already have at least \( |T| \) chains in \( CD' \) before adding \( C \), so we have to find different chains. To do this, we will (roughly speaking) “split” the set \( A \) and relation \( R \) into two parts, obtain chain decompositions on each part by the induction hypothesis, then “sew” these together to obtain a single chain decomposition of \( R \) on \( A \) of the right size, specifically of size \( |T| \).

First, we observe that, since any antichain of \( R' \) on \( A' \) is an antichain of \( R \) on \( A \), we have \( |T'| \leq |T| \). We already have \( |T'| \geq |T| \) in this case, so \( |T'| = |T| \). Thus, \( T' \) is a
3. A tournament graph is a directed graph where exactly one of \((u,v)\) or \((v,u)\) is in \(E\) for every pair of distinct vertices \(u,v \in V\). A champion of the graph is a vertex from which every other vertex is reachable by a path of length at most two from the champion. That is, every other vertex is an out-neighbor of the champion, or it is the out-neighbor of an out-neighbor of the champion. Prove that any vertex in \(G\) with largest out-degree is a champion.

Solution: Suppose for sake of contradiction a vertex \(v\) with maximum out-degree is not a champion, and let \(X\) be the set of out-neighbors of \(v\). Since \(u\) is not a champion, there is some vertex \(v\) that \(u\) cannot reach via paths of length at most two; in particular, \(v\) is not in \(X\), and \(v\) is not an out-neighbor of any vertex in \(Y\). In other words, \((x,v) \notin E\) for every \(x \in X \cup \{u\}\). Since \(G\) is a tournament graph, we have \((v,x) \in E\) for every \(x \in X \cup \{u\}\). It follows that the out-degree of \(v\) is at least one more than \(|X|\), the out-degree of \(u\), which is a contradiction.