1. In lecture, we proved that for all \( x \in \mathbb{R} \) (except \( x = 1 \)) and \( n \in \mathbb{N} \), the sum \( S_{n,x} = 1 + x + x^2 + x^3 + \ldots + x^n \) has a closed form

\[
S_{n,x} = \frac{1 - x^{n+1}}{1 - x}.
\]

Give a closed form for the sum \( T_{n,x} = x + 2x^2 + 3x^3 + \ldots + nx^n \) for all \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \).

**Solution:** Similar to the how we obtained the closed form of \( S \) in lecture, we consider subtracting \( xT_{n,x} = x^2 + 2x^3 + 3x^4 + \ldots + nx^{n+1} \) from \( T_{n,x} \) to obtain

\[
T_{n,x} - xT_{n,x} = (1 - x)T_{n,x} = x + x^2 + x^3 + \ldots + x^n - nx^{n+1}.
\]

Note that the right-hand side contains \( x + x^2 + x^3 + \ldots + x^n = S_{n,x} - 1 \), so we have

\[
(1 - x)T_{n,x} = S_{n,x} - 1 - nx^{n+1} = \frac{1 - x^{n+1}}{1 - x} - 1 - nx^{n+1}.
\]

Solving for \( T_{n,x} \), we get

\[
T_{n,x} = \frac{1 - x^{n+1} - 1 - nx^{n+1}}{1 - x} = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2}.
\]

2. Approximate \( \sum_{i=1}^{n} i^{2.3} \) using integrals.

**Solution:** Let \( f(x) = x^{2.3} \) which is clearly increasing. From lecture, we showed how to approximate sums by integrals (the "opposite" as you likely saw in a calculus course — approximating integrals with sums). Specifically, we showed that for a sum of the form \( \sum_{i=1}^{n} f(i) \) for a non-decreasing function, we have\(^1\)

\[
f(1) + \int_{1}^{n} f(x) \, dx \leq \sum_{i=1}^{n} f(i) \leq \int_{1}^{n} f(x) \, dx + f(n).
\]

Plugging in our function \( f(x) = x^{2.3}, f(1) = 1^{2.3} = 1 \) and \( f(n) = n^{2.3} \) into these inequalities, we have

\[
1 + \int_{1}^{n} x^{2.3} \, dx \leq \sum_{i=1}^{n} i^{2.3} \leq \int_{1}^{n} x^{2.3} \, dx + n^{2.3}
\]

. After solving the integral,

\[
\int_{1}^{n} f(x) \, dx = \left. \frac{1}{3.3} x^{3.3} \right|_{1}^{n} = \frac{1}{3.3} (n^{3.3} - 1),
\]

\(^1\)When \( f \) is non-increasing, we have the same inequalities except \( f(1) \) and \( f(n) \) are swapped.
and substituting it in, we conclude the following:

\[
1 + \frac{1}{3.3} (n^{3.3} - 1) \leq \sum_{i=1}^{n} x^{2.3} \leq \frac{1}{3.3} (n^{3.3} - 1) + n^{2.3}.
\]

3. In lecture, we argued that every finite binary string \( s \in \{0, 1\}^* \) both represents a program, denoted by \( P_s \), and the string \( s \) itself which represents a possible input. We denote by \( P_s \) the program represented by string \( s \in \{0, 1\}^* \). When a string \( s \) is input to a program \( P \), the execution of \( P \) must either halt on input \( s \), or it never halts (e.g. reaches an infinite loop). We define the language of a program \( P \) as \( \text{lang}(P) = \{ s \in \{0, 1\}^* \mid P \text{ halts on input } s \} \).

Recall the language \( L = \{ s \in \{0, 1\}^* \mid P_s \text{ does not halt on input } s \} \). We proved there is no program \( P \) such that \( \text{lang}(P) = L \); that is, there is no program \( P \) such that, for all \( s \in \{0, 1\}^* \), \( P \) does not halt on \( s \) if and only if \( P_s \) halts on \( s \). Unfortunately, this further implies there exists no computer program that can correctly determine whether another computer program will halt or not.

Prove (again) the unfortunate fact that there exists no program \( P \) such that \( \text{lang}(P) = L \) using diagonalization.

**Solution:** Since \( \{0, 1\}^* \) is countably infinite, we can impose an ordering all elements of the set; let \( s_i \) be the \( i \)th string in this ordering. Formally, let \( f \) be a bijection from \( \mathbb{N} \) to \( \{0, 1\}^* \), then let \( s_i = f(i) \) for all \( i \in \mathbb{N} \). Denote by \( s(j) \) the \( j \)th bit (0 or 1) in a string \( s \in \{0, 1\}^* \). For simplicity, we also define \( P_i = P_{s_i} \) for all \( i \in \mathbb{N} \).

Now define \( M \) to be the infinite 2d matrix such that, for all \( i, j \in \mathbb{N} \), \( M_{ij} \) is 1 if \( P_i \) halts on input \( s_j \), and 0 otherwise. Roughly speaking, the rows of \( M \) describe the halting behavior of each program.

Now consider the infinite binary string \( s^* = M_{1,1} M_{2,2} M_{3,3} \ldots \) obtained by “flipping” each bit in the infinite diagonal of \( M \); in other words, let \( s^* \in \{0, 1\}^* \) where \( s^*(j) = \overline{M_{j,j}} \) for all \( j \in \mathbb{N} \).

By construction, \( s^* \) is not equal to any row of \( M \); indeed, it differs from any row \( r_i \) (at least) in the \( i \)th position. If it was a row of \( M \) for some program \( P^* \), what would \( \text{lang}(P^*) \) be? By the definition of \( M \) and \( s^* \), we have, for any \( i \in \mathbb{N} \):

\[
P^* \text{ halts on input } s_i \iff s^*(i) = 1 \\
\iff M_{i,i} = 1 \\
\iff M_{i,i} = 0 \\
\iff P_i \text{ does not halt on input } s_i
\]

In other words, \( \text{lang}(P^*) = \{ s_i \in \{0, 1\}^* \mid P_i \text{ halts on input } s_i \} = L \)! Thus, not only have we shown that the unfortunate fact there is some language (represented by \( s^* \)) that no program has, we’ve shown (again) that there is no program whose language is \( L \).
4. Prove $\mathbb{R}^2 bij \mathbb{R}$ via the Schröder-Bernstein theorem. That is, show that $\mathbb{R}^2 inj \mathbb{R}$ and $\mathbb{R} inj \mathbb{R}^2$, then apply the theorem.

**Solution:** Note that following solution diverges slightly from the one presented in recitation, and primarily for simplicity and to bring the main idea forward.

From the homework 10 handout, we know $\mathbb{R} bij (0, 1)$. Let $f : \mathbb{R} \to (0, 1)$ be a bijection, then define $g : \mathbb{R} \to (0, 1)^2$ such that $g(x, y) = (f(x), f(y))$. It is easy to verify that $g$ is a bijection since $f$ is a bijection, hence $\mathbb{R}^2 bij (0, 1)^2$. Thus, if we show $(0, 1)^2 bij (0, 1)$, we can conclude $\mathbb{R}^2 bij \mathbb{R}$ by transitivity; that is, $\mathbb{R}^2 bij (0, 1)^2$, $(0, 1)^2 bij (0, 1)$, and $(0, 1) bij \mathbb{R}$ implies $\mathbb{R}^2 bij \mathbb{R}$.

The Schröder-Bernstein theorem implies that if $A inj B$ and $B inj A$ for sets $A, B$, then $A bij B$. Thus, we will complete the proof by showing $(0, 1) inj (0, 1)^2$ and $(0, 1)^2 inj (0, 1)$.

First, see that $(0, 1) inj (0, 1)^2$ by defining $f : (0, 1) \to (0, 1)^2$ where $f(x) = (x, \frac{1}{2})$ for all $x \in (0, 1)$. Clearly $f(x) = f(y) \implies (x, \frac{1}{2}) = (y, \frac{1}{2}) \implies x = y$ for any $x, y \in \mathbb{R}$, so $f$ is an injective function.

To see that $(0, 1)^2 inj (0, 1)$, we define $d(x)$ be (any one of the) infinite decimal expansion(s) for $x$. For example:

- If $x = \frac{230}{100}$, then $d(x) = .230$
- If $x = \frac{230}{300}$, then $d(x) = .766$
- If $x = \sqrt{\frac{230}{300}}$, then $d(x) = .875595\ldots$

Then we define $f(x, y)$ to be the *interleaving* of $d(x)$ and $d(y)$ as follows. Consider any $x, y \in (0, 1)$ where

$$d(x) = .x_1x_2x_3\ldots$$
$$d(y) = .y_1y_2y_3\ldots$$

We define

$$f(x, y) = x_1y_1x_2y_2x_3y_3\ldots$$

For example, $f(\sqrt{\frac{230}{300}}, \frac{230}{300}) = f(.230595, .230)$ is defined to be $.827350509050\ldots$.

To finish the proof, it remains to argue $f$ is an injection. Let $a, b, c, d$ be arbitrary real numbers such that either $a \neq c$ or $b \neq d$. We will show $f(a, b) \neq f(c, d)$ which implies $f$ is an injection. Without loss of generality, $a \neq c$. Then $a_i \neq c_i$ for some $i \in \mathbb{N}$. The definition of $f$ places the $i$th digit of $a_i$ at the index in $f(a, b)$ as the $i$th digit of $c_i$ in $f(a, b)$, so $f(a, b) \neq f(c, d)$. The case where $a = c$ and $b \neq d$ is similar. We conclude $(0, 1)^2 inj (0, 1)$ which implies $\mathbb{R}^2 inj \mathbb{R}$ following the arguments at the beginning of this solution.

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Some nonzero rational numbers have two infinite representations, e.g. $\frac{1}{2} = .5\overline{0} = .4\overline{9}$. For our purposes, we can choose either arbitrarily when defining our injection.