- **Bipartite Graphs**
  - used as abstractions for relations of two types of objects.

- $G = (U, V, E) \quad E \subseteq \{ (u, v) \mid u \in U, v \in V \}$
  - every edge connects one vertex in $U$ with another vertex in $V$.

- $|U| = n_1, \quad |V| = n_2, \quad |M| = m$

- a matching $M$ of a bipartite graph is a subset of edges such that no two edges in $M$ share a vertex.

\[
\{ (1,1), (3,2), (2,3) \} \text{ is matching}
\]

\[
\{ (2,2), (2,3) \} \text{ is not a matching}
\]

- size of a matching $M$ is the number of edges in $M$
- maximum matching is a matching with maximum number of edges.

- for a bipartite graph $G$ and a matching $M$
  - call a vertex "matched" if the vertex is adjacent to an edge in $M$
    - $(1, 2, a, b$ matched, $3, c$ are unmatched)
  - call an edge $e$ matched if $e \in M$, otherwise $e$ is unmatched
  - augmenting path: path connecting two unmatched vertices, the edges alternate between unmatched edge and matched edge.

\[
(2 \quad 1 \quad b \quad 3 \quad a) \text{ is an augmenting path}
\]
and matched edge.

\[(3, b) (b, 2) (2, c)\] is an augmenting path.

\[(3, b) (b, 1) (1, c)\] is not an augmenting path.

Fact: length of augmenting path is always odd.

First and last edges of the augmenting path are unpaired.

- XOR operation \(\oplus\)

\[\text{if } x \text{ and } y \text{ are } 0, 1\] \(x \oplus y\) is equal to 1

\[\text{if and only if } x \neq y\]

\[\text{(if only one of } x, y \text{ is equal to 1)}\]

\[\text{only in } M\]

\[\text{only in } P\]

Matching \(\text{augmenting path}\) \(\text{matching with one more edge}\).

- Using DFS to find augmenting path
- Proof of correctness
  
  Assume towards contradiction that M is not a maximum matching.
  
  Let \( M' \) be a maximum matching
  
  \( |M'| > |M| \)
  
  Consider \( M' \oplus M \)
  
  Claim: \( M' \oplus M \) is going to have only paths or cycles.
  
  Idea: in \( M' \oplus M \), every vertex has degree at most 2.

- For a cycle: it contains the same \# of edges in \( M, M' \)
- For a path: one of the matching has one more edge of odd length

Know: \( |M'| > |M| \), so there must be a path of odd length where first and last edge are \( M' \); this path is an augmenting path for \( M \) and this contradicts the assumption that \( M \) does not
Assumption that $M$ does not have any augmenting path.

3 $\rightarrow$ c $\rightarrow$ c
4 $\rightarrow$ d $\rightarrow$ od
5 $\rightarrow$ of $\rightarrow$ og
6 $\rightarrow$ M $\rightarrow$ $M'$ $\rightarrow$ $M \oplus M'$