- Primal and dual

Primal

\[
\begin{align*}
\text{min } & (2 \quad -3 \quad 1) \cdot x \\
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -2 \\
-1 & -1 & -1 
\end{bmatrix} & \geq \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

Dual

\[
\begin{align*}
\text{max } & (1 \quad 2 \quad -7) \cdot y \\
\begin{bmatrix}
1 & 0 & -1 \\
-1 & 1 & -1 \\
0 & -2 & -1 
\end{bmatrix} & \leq \\
y_1, y_2, y_3 & \geq 0
\end{align*}
\]

\[(y_1, y_2, y_3) = (2.5, 0, 0.5)\] dual optimal solution

\[y_1 \bar{x}_1 (x_1 - x_2 \geq 1) + y_2 \bar{x}_2 (x_2 - 2x_3 \geq 2) + y_3 \bar{x}_3 (x_1 - x_2 - x_3 \geq -7)\]

\[\Rightarrow \\
2x_1 - 3x_2 - \frac{1}{2} x_3 \geq -1
\]

\[2x_1 - 3x_2 + x_3 \geq 2x_1 - 2x_2 - \frac{1}{2} x_3 \geq -1\]

\[y_1 - y_3 - y_1 + y_2 - y_3 - 2y_2 - y_3
\]

\[(x_1, x_2, x_3) = (4, 3, 0)\] primal optimal solution

\[4 \bar{x}_1 (y_1 - y_3 \leq 2) + 3 (-y_1 + y_2 - y_3 \leq -3)\]

\[\Rightarrow y_1 + 3y_2 - 7y_3 \leq -1
\]

\[y_1 + 2y_2 - 7y_3 \leq y_1 + 3y_2 - 7y_3 \leq -1\]

- Complementary slackness
- let \( x \) and \( y \) be optimal solutions of primal and dual LP.
- call a constraint tight if the LHS is exactly equal to the RHS.
- if \( i \)-th constraint in primal is not tight, then the \( i \)-th variable of the dual is equal to 0.
- i-th variable of the dual is equal to 0
- i-th constraints in dual is not tight, then the i-th variable of the primal is equal to 0.

- primal slack \( i \) = LHS of primal constraint \( i \) - RHS of primal constraint \( i \)
  
  e.g. primal slack \( i \) = \( x_1 - x_2 - 1 \)
  
  \[
  \begin{pmatrix}
  \text{primal slack } \( i \) \\
  \end{pmatrix} \times y_i = 0
  \]

- Simplex algorithm
  - basic feasible solution
    - geometric: vertex of the feasible region
    - linear algebraic: feasible solution where there are \( n \) tight constraints with linearly independent coefficients.

  e.g. \((4,3,6)\) is a basic feasible solution because it is a solution of

  \[
  \begin{align*}
  x_1 - x_2 &= 1 \\
  -x_1 - x_2 - x_3 &= -7 \\
  x_3 &= 0
  \end{align*}
  \]

  \[
  \begin{pmatrix}
  x_1 - x_2 \\
  -x_1 - x_2 - x_3 \\
  x_3
  \end{pmatrix} \times y_i = 0
  \]

- ellipsoid algorithm
  - main idea: maintain an ellipsoid
    - check if the center is feasible
    - if not, find the violating constraint
if not, find the violating constraint and cut the ellipsoid in half.