This recitation is to review the material from sections 1.1-1.5 in the course textbook as covered in lecture 1. Specifically, we will prove the truth of some example propositions. First, some reminders:

1: Everyone should have access to Sakai, Piazza, and Gradescope. Please contact the course staff TODAY if you do not have access to one of these platforms.

2: Lecture notes will be available on the course website with two days of lecture (Lecture 1’s are available now).

3: Homework 1 will be released on Monday, 1/14.

4: Office hours will also start on Monday, 1/14.

5: All feedback is welcome and appreciated. Let us know via email or Piazza!

Recall the two approaches described in lecture to prove an implication — a proposition of the form “If $P$, then $Q$,” “$P$ implies $Q$,” or “$Q$ if $P$.” $P$ is called the antecedent, and $Q$ is called the consequent.

In this recitation, we will implicitly assume all numbers are positive integers.

1. (Review from Lecture 1) Show that the following proposition is True: If $x$ an even and prime integer, then $x = 2$.

   **Discussion:** To prove an implication True, one approach is to first assume the antecedent, then show (through a sequence of steps) that the consequent follows.

   Another approach is to prove its contrapositive True. Recall that the contrapositive of an implication “If $P$, then $Q$” is “If not $Q$, then not $P$.” To prove a proposition via the contrapositive, state the contrapositive, then proceed with a direct proof for that implication.

   **Solution:** We will first show the proposition is True via direct proof, then again by proving the contrapositive.

   *Direct proof:* Assume $x$ is a prime number. Since $x$ is prime, $x$ is not divisible by a number smaller than $x$ (other than 1). Since $x$ is even, $x$ is divisible by 2. It follows that $x$ is a number divisible by only 2 and 1, and thus $x = 2$. Therefore, the implication is True.

   *Proof by contrapositive:* The contrapositive is as follows: If $x \neq 2$, then $x$ is not even or $x$ is not prime. Assume $x$ is an integer not equal to 2. There are two cases for $x$: $x$ is even, or $x$ is not even.

   - Assume $x$ is even. Then $x$ is divisible by 2. Since $x$ is not 2, it must be that $x$ is strictly greater than 2, and thus $x$ is divisible by a number less than $x$ and other than 1. This implies $x$ is not prime. Then the consequent, $x$ is not even or not prime, follows and thus the contrapositive is True in this case.
   - Assume $x$ is not even. Then clearly the consequent, $x$ is not even or not prime, follows thus the contrapositive is True in this case.

   Since the contrapositive is True in both cases, the original implication is True.

2. Show that the following proposition is True with a direct proof: If $x$ is an even and square integer, then $x$ is divisible by 4.
Solution: Let $x$ be an arbitrary even and square integer. Since $x$ is square, then $x = k^2$ for some positive integer $k$. Since $x$ is even, $k^2$ is even, which implies $k$ is even. This implies $k = 2\ell$ for some positive integer $\ell$, so $x = k^2 = (2\ell)^2 = 4\ell^2$ which is divisible by 4. We conclude that the proposition is True.

3. Show that the following proposition is True by proving its contrapositive: If $x^2$ is divisible by 4, then $x$ is even.

Solution: The contrapositive is as follows: If $x$ is odd, then $x^2$ is not divisible by 4. Assume $x$ is an odd positive integer. Since $x$ is odd, $x^2$ is odd. Then $x^2$ is not divisible by 4, and thus the contrapositive is True.

4. Show that the following proposition is False: If $x$ is even, square, and greater than 4, then $x$ is divisible by 8.

Discussion: Recall that to prove an implication is false, we must exhibit an $x$ for which the antecedent is True, but the consequent is False. Such an $x$ is called a counterexample. For this particular problem, this means we must identify $x$ such that $x$ that is even, square, and greater than 4, but not divisible by 8.

Solution: Let $x = 36$. $x$ is even, square, and greater than 4, but $x$ is not divisible by 8. Thus, the proposition is False.

5. Show that the following proposition is False: If $x^2$ is divisible by 1323, then $x$ is divisible by 1323.

Discussion: For this proposition to be false, there must exist an $x$ such that $x^2$ is divisible by 1323, but $x$ is not divisible by 1323. Our goal is to identify such an $x$.

Consider the prime factorization of 1323 = $3^3 \cdot 7^2$. Any counterexample $x$ must be such that $x$ has less than three multiples of 3 in its prime factorization or less than two multiples of 7 in its prime factorization (otherwise $x$ would be divisible by 1323). Furthermore, a counterexample $x$ must be so that the prime factorization of $x^2$ has at least three multiples of 3 and two multiples of 7 (so that $x^2$ is divisible by 1323).

Solution: Let $x = 3^2 \cdot 7^1 = 63$. Then $x^2 = 3969$ which is divisible by 1323. However, $x$ is not divisible by 1323. Thus, $x = 63$ is a counterexample to the given proposition, so it is False.

(Note that in the previous problem, 63 is only one of infinitely many counterexamples, but it is the smallest positive counterexample. Can you prove these two propositions?)