Game Theory

Instructor: Vincent Conitzer
Penalty kick example

Is this a "rational" outcome? If not, what is?
Rock-paper-scissors

Row player aka. player 1 chooses a row

Column player aka. player 2 (simultaneously) chooses a column

A row or column is called an action or (pure) strategy

Row player’s utility is always listed first, column player’s second

Zero-sum game: the utilities in each entry sum to 0 (or a constant)
Three-player game would be a 3D table with 3 utilities per entry, etc.

<table>
<thead>
<tr>
<th></th>
<th>0, 0</th>
<th>-1, 1</th>
<th>1, -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, -1</td>
<td>0, 0</td>
<td>-1, 1</td>
<td></td>
</tr>
<tr>
<td>-1, 1</td>
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<td>0, 0</td>
<td></td>
</tr>
</tbody>
</table>
A poker-like game

```
<table>
<thead>
<tr>
<th></th>
<th>cc</th>
<th>cf</th>
<th>fc</th>
<th>ff</th>
</tr>
</thead>
<tbody>
<tr>
<td>rr</td>
<td>0, 0</td>
<td>0, 0</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td>rc</td>
<td>.5, -.5</td>
<td>1.5, -1.5</td>
<td>0, 0</td>
<td>1, -1</td>
</tr>
<tr>
<td>cr</td>
<td>-.5, .5</td>
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<td>1, -1</td>
<td>1, -1</td>
</tr>
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“Chicken”

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die

<table>
<thead>
<tr>
<th></th>
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<th>S</th>
</tr>
</thead>
<tbody>
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<td>D</td>
<td>0, 0</td>
<td>-1, 1</td>
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<td>S</td>
<td>1, -1</td>
<td>-5, -5</td>
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not zero-sum
“2/3 of the average” game

- Everyone writes down a number between 0 and 100
- Person closest to 2/3 of the average wins

Example:
- A says 50
- B says 10
- C says 90
- Average(50, 10, 90) = 50
- 2/3 of average = 33.33
- A is closest (|50-33.33| = 16.67), so A wins
MICKEY: All right, rock beats paper!
(Mickey smacks Kramer's hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.

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Dominance

- Player i’s strategy $s_i$ strictly dominates $s_i'$ if
  - for any $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

- $s_i$ weakly dominates $s_i'$ if
  - for any $s_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$; and
  - for some $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

\[-i = \text{“the player(s) other than } i\text{”}\]
Prisoner’s Dilemma

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 1 year reduction
  - If only one confesses, that one gets 3 years reduction

<table>
<thead>
<tr>
<th></th>
<th>confess</th>
<th>don’t confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>confess</td>
<td>-2, -2</td>
<td>0, -3</td>
</tr>
<tr>
<td>don’t confess</td>
<td>-3, 0</td>
<td>-1, -1</td>
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</table>
“Should I buy an SUV?”

<table>
<thead>
<tr>
<th>Cost</th>
<th>Purchase + Gas Cost</th>
<th>Accident Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10, -10</td>
<td>cost: 5</td>
<td>-7, -11</td>
</tr>
<tr>
<td>-11, -7</td>
<td>cost: 3</td>
<td>-8, -8</td>
</tr>
<tr>
<td>-11, -11</td>
<td>cost: 8</td>
<td></td>
</tr>
<tr>
<td>-10, -10</td>
<td>cost: 5</td>
<td></td>
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<td>cost: 5</td>
<td></td>
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Back to the poker-like game

1 gets King
player 1
raise
1 gets Jack
check
raise
player 1
check
player 2
fold
call
fold
call
fold

"nature"

Player 1

Player 2

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Iterated dominance

• Iterated dominance: remove (strictly/weakly) dominated strategy, repeat

• Iterated strict dominance on Seinfeld’s RPS:

\[
\begin{array}{ccc}
0, 0 & 1, -1 & 1, -1 \\
-1, 1 & 0, 0 & -1, 1 \\
-1, 1 & 1, -1 & 0, 0 \\
\end{array}
\]
“2/3 of the average” game revisited

\[
\text{dominated after removal of (originally) dominated strategies}
\]
Mixed strategies

- **Mixed strategy** for player $i =$ **probability distribution** over player $i$’s (pure) strategies

- E.g. $1/3$, $1/3$, $1/3$

- Example of dominance by a mixed strategy:
Nash equilibrium [Nash 1950]

- A profile (= strategy for each player) so that no player wants to deviate

\[
\begin{array}{c|cc}
 & D & S \\
\hline
D & 0, 0 & -1, 1 \\
S & 1, -1 & -5, -5 \\
\end{array}
\]

- This game has another Nash equilibrium in mixed strategies…
Rock-paper-scissors

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- Any pure-strategy Nash equilibria?
- But it has a **mixed-strategy Nash equilibrium**: Both players put probability 1/3 on each action
- If the other player does this, every action will give you expected utility 0
  - Might as well randomize
Nash equilibria of “chicken”…

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- Is there a Nash equilibrium that uses mixed strategies? Say, where player 1 uses a mixed strategy?
- If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses.
- So we need to make player 1 indifferent between D and S.
- Player 1’s utility for playing D = \(-p^c_S\)
- Player 1’s utility for playing S = \(p^c_D - 5p^c_S = 1 - 6p^c_S\)
- So we need \(-p^c_S = 1 - 6p^c_S\) which means \(p^c_S = 1/5\)
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
  - People may die! Expected utility -1/5 for each player
The presentation game

<table>
<thead>
<tr>
<th>Pay attention (A)</th>
<th>Do not pay attention (NA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Put effort into presentation (E)</td>
<td>2, 2</td>
</tr>
<tr>
<td>Do not put effort into presentation (NE)</td>
<td>-1, 0</td>
</tr>
</tbody>
</table>

- Pure-strategy Nash equilibria: (E, A), (NE, NA)
- Mixed-strategy Nash equilibrium:
  \(((4/5 \text{ E}, 1/5 \text{ NE}), (1/10 \text{ A}, 9/10 \text{ NA}))\)
  - Utility -7/10 for presenter, 0 for audience
Back to the poker-like game, again

- To make player 1 indifferent between rr and rc, we need:
  utility for rr = 0*P(cc)+1*(1-P(cc)) = .5*P(cc)+0*(1-P(cc)) = utility for rc
  That is, P(cc) = 2/3

- To make player 2 indifferent between cc and fc, we need:
  utility for cc = 0*P(rr)+(-.5)*(1-P(rr)) = -1*P(rr)+0*(1-P(rr)) = utility for fc
  That is, P(rr) = 1/3
Real-world security applications

Airport security
Where should checkpoints, canine units, etc. be deployed?

Federal Air Marshals
Which flights get a FAM?

US Coast Guard
Which patrol routes should be followed?

Wildlife Protection
Where to patrol to catch poachers or find their snares?