Penalty kick example

Is this a "rational" outcome? If not, what is?
Real-world security applications

Airport security
Where should checkpoints, canine units, etc. be deployed?

Federal Air Marshals
Which flights get a FAM?

US Coast Guard
Which patrol routes should be followed?

Wildlife Protection
Where to patrol to catch poachers or find their snares?

Milind Tambe’s TEAMCORE group (USC)
Rock-paper-scissors

Column player aka. player 2 (simultaneously) chooses a column

Row player aka. player 1 chooses a row

A row or column is called an action or (pure) strategy

Row player’s utility is always listed first, column player’s second

Zero-sum game: the utilities in each entry sum to 0 (or a constant)
Three-player game would be a 3D table with 3 utilities per entry, etc.

<table>
<thead>
<tr>
<th></th>
<th>0, 0</th>
<th>-1, 1</th>
<th>1, -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, -1</td>
<td>0, 0</td>
<td>-1, 1</td>
<td></td>
</tr>
<tr>
<td>-1, 1</td>
<td>1, -1</td>
<td>0, 0</td>
<td></td>
</tr>
</tbody>
</table>
A poker-like game

```
1 gets King
player 1
raise
```

```
player 1
check
raise
```

```
player 2
```

```
call
fold
call
fold
call
fold
```

```
1 gets Jack
```

```
player 1
check
```

```
player 2
```

```
call
fold
call
fold
call
fold
```

```
"nature"
```

<table>
<thead>
<tr>
<th></th>
<th>cc</th>
<th>cf</th>
<th>fc</th>
<th>ff</th>
</tr>
</thead>
<tbody>
<tr>
<td>rr</td>
<td>0, 0</td>
<td>0, 0</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td>rc</td>
<td>.5, -.5</td>
<td>1.5, -1.5</td>
<td>0, 0</td>
<td>1, -1</td>
</tr>
<tr>
<td>cr</td>
<td>-.5, .5</td>
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<td>0, 0</td>
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</tbody>
</table>
“Chicken”

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die

<table>
<thead>
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<th></th>
<th>S</th>
<th>D</th>
</tr>
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<tbody>
<tr>
<td>D</td>
<td>0, 0</td>
<td>-1, 1</td>
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<td>1, -1</td>
<td>-5, -5</td>
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not zero-sum
“2/3 of the average” game

• Everyone writes down a number between 0 and 100
• Person closest to 2/3 of the average wins
• Example:
  – A says 50
  – B says 10
  – C says 90
  – Average(50, 10, 90) = 50
  – 2/3 of average = 33.33
  – A is closest (|50-33.33| = 16.67), so A wins
MICKEY: All right, rock beats paper!
(Mickey smacks Kramer's hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.
Dominance

- Player i’s strategy $s_i$ strictly dominates $s_i’$ if
  - for any $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i’, s_{-i})$

- $s_i$ weakly dominates $s_i’$ if
  - for any $s_{-i}$, $u_i(s_i, s_{-i}) ≥ u_i(s_i’, s_{-i})$; and
  - for some $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i’, s_{-i})$

\[-i = “the player(s) other than i”\]
Prisoner’s Dilemma

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 1 year reduction
  - If only one confesses, that one gets 3 years reduction

<table>
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<tr>
<th></th>
<th>confess</th>
<th>don’t confess</th>
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</thead>
<tbody>
<tr>
<td>confess</td>
<td>-2, -2</td>
<td>0, -3</td>
</tr>
<tr>
<td>don’t confess</td>
<td>-3, 0</td>
<td>-1, -1</td>
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</table>
“Should I buy an SUV?”

**purchasing + gas cost**

- SUV: cost: 5
- Sedan: cost: 3

**accident cost**

- SUV: cost: 5 (2 collisions)
- Sedan: cost: 8 (1 collision)
- Other Sedan: cost: 2 (no collisions)
- Other SUV: cost: 5 (1 collision)

<table>
<thead>
<tr>
<th></th>
<th>-10, -10</th>
<th>-7, -11</th>
<th>-11, -7</th>
<th>-8, -8</th>
</tr>
</thead>
</table>

**Decision:**

- SUV: -10, -10 (high cost, no accidents)
- Sedan: -7, -11 (low cost, high accidents)
- Other Sedan: -11, -7 (medium cost, medium accidents)
- Other SUV: -8, -8 (low cost, high accidents)
Back to the poker-like game

1 gets King
player 1
raise
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raise
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player 2
call
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1 gets Jack

"nature"

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</table>
**Iterated dominance**

- Iterated dominance: remove (strictly/weakly) dominated strategy, repeat
- Iterated strict dominance on Seinfeld’s RPS:
"2/3 of the average" game revisited

\[
(2/3)^n \times 100
\]

... dominated

\[
(2/3)^n \times (2/3)^n \times 100
\]

... dominated after removal of (originally) dominated strategies
Mixed strategies

- Mixed strategy for player i = probability distribution over player i’s (pure) strategies
- E.g. \( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \)
- Example of dominance by a mixed strategy:

\[
\begin{array}{cc}
3, 0 & 0, 0 \\
0, 0 & 3, 0 \\
1, 0 & 1, 0 \\
\end{array}
\]
Nash equilibrium [Nash 1950]

- A profile (= strategy for each player) so that no player wants to deviate

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- This game has another Nash equilibrium in mixed strategies…
### Rock-paper-scissors

![Images of rock, paper, and scissors]

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<tr>
<td>-1, 1</td>
<td>1, -1</td>
<td>0, 0</td>
<td></td>
</tr>
</tbody>
</table>

- Any pure-strategy Nash equilibria?
- But it has a **mixed-strategy Nash equilibrium**: Both players put probability 1/3 on each action.
- If the other player does this, every action will give you expected utility 0
  - Might as well randomize
Nash equilibria of “chicken”…

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</tr>
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- Is there a Nash equilibrium that uses mixed strategies? Say, where player 1 uses a mixed strategy?
- If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses
- So we need to make player 1 indifferent between D and S
- Player 1’s utility for playing D = \(-p_c^S\)
- Player 1’s utility for playing S = \(p_c^D - 5p_c^S = 1 - 6p_c^S\)
- So we need \(-p_c^S = 1 - 6p_c^S\) which means \(p_c^S = 1/5\)
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
  - People may die! Expected utility \(-1/5\) for each player
The presentation game

<table>
<thead>
<tr>
<th>Pay attention (A)</th>
<th>Do not pay attention (NA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Put effort into presentation (E)</td>
<td>2, 2</td>
</tr>
<tr>
<td>Do not put effort into presentation (NE)</td>
<td>-7, -8</td>
</tr>
</tbody>
</table>

- Pure-strategy Nash equilibria: (E, A), (NE, NA)
- Mixed-strategy Nash equilibrium:
  \[ ((4/5 \text{ E, } 1/5 \text{ NE}), (1/10 \text{ A, } 9/10 \text{ NA})) \]
  - Utility -7/10 for presenter, 0 for audience
Back to the poker-like game, again

• To make player 1 indifferent between rr and rc, we need:
  utility for rr = 0*P(cc)+1*(1-P(cc)) = .5*P(cc)+0*(1-P(cc)) = utility for rc
  That is, P(cc) = 2/3
• To make player 2 indifferent between cc and fc, we need:
  utility for cc = 0*P(rr)+(-.5)*(1-P(rr)) = -1*P(rr)+0*(1-P(rr)) = utility for fc
  That is, P(rr) = 1/3
Commitment

• Consider the following (normal-form) game:

<table>
<thead>
<tr>
<th></th>
<th>2, 1</th>
<th>4, 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 0</td>
<td>3, 1</td>
<td></td>
</tr>
</tbody>
</table>

• How should this game be played?
• Now suppose the game is played as follows:
  – Player 1 **commits** to playing one of the rows,
  – Player 2 observes the commitment and then chooses a column

• What is the optimal strategy for player 1?
• What if 1 can commit to a **mixed** strategy?
Commitment as an extensive-form game

- For the case of committing to a pure strategy:

  ![Game Tree](image)

  - Player 1
    - Up
      - Player 2
        - Left: 2, 1
        - Right: 4, 0
    - Down
      - Player 2
        - Left: 1, 0
        - Right: 3, 1
Commitment as an extensive-form game

- For the case of committing to a mixed strategy:

- Infinite-size game; computationally impractical to reason with the extensive form here
Solving for the optimal mixed strategy to commit to

[Conitzer & Sandholm 2006, von Stengel & Zamir 2010]

• For every column c separately, we will solve separately for the best mixed row strategy (defined by \( p_r \)) that induces player 2 to play c

• maximize \( \Sigma_r p_r u_1(r, c) \)

• subject to
  
  for any \( c' \), \( \Sigma_r p_r u_2(r, c) \geq \Sigma_r p_r u_2(r, c') \)
  
  \( \Sigma_r p_r = 1 \)

• (May be infeasible, e.g., if c is strictly dominated)

• Pick the c that is best for player 1
## Visualization

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0,1</td>
<td>1,0</td>
<td>0,0</td>
</tr>
<tr>
<td>M</td>
<td>4,0</td>
<td>0,1</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>1,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

(1,0,0) = U

(0,1,0) = M

(0,0,1) = D