Artificial Intelligence

Markov processes and Hidden Markov Models (HMMs)

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Motivation

- The Bayes nets we considered so far were static: they referred to a single point in time
  - E.g., medical diagnosis
- Agent needs to model how the world evolves
  - Speech recognition software needs to process speech over time
  - Artificially intelligent software assistant needs to keep track of user’s intentions over time
  - … … …
Markov processes

• We have time periods $t = 0, 1, 2, \ldots$
• In each period $t$, the world is in a certain state $S_t$
• The **Markov assumption**: given $S_t$, $S_{t+1}$ is independent of all $S_i$ with $i < t$
  
  - $P(S_{t+1} \mid S_1, S_2, \ldots, S_t) = P(S_{t+1} \mid S_t)$
  - Given the current state, history tells us nothing more about the future

  \[ S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow \ldots \rightarrow S_t \rightarrow \ldots \]

• Typically, all the CPTs are the same:
• For all $t$, $P(S_{t+1} = j \mid S_t = i) = a_{ij}$ (**stationarity assumption**)
Weather example

- $S_t$ is one of $\{s, c, r\}$ (sun, cloudy, rain)
- Transition probabilities:

- Also need to specify an initial distribution $P(S_0)$
- Throughout, assume $P(S_0 = s) = 1$
Weather example...

- What is the probability that it rains two days from now?  \( P(S_2 = r) \)
- \( P(S_2 = r) = P(S_2 = r, S_1 = r) + P(S_2 = r, S_1 = s) + P(S_2 = r, S_1 = c) = .1 \times .3 + .6 \times .1 + .3 \times .3 = .18 \)
What is the probability that it rains three days from now?

Computationally inefficient way: \( P(S_3 = r) = P(S_3 = r, S_2 = r, S_1 = r) + P(S_3 = r, S_2 = r, S_1 = s) + \ldots \)

For \( n \) periods into the future, need to sum over \( 3^{n-1} \) paths
More efficient:

\[ P(S_3 = r) = P(S_3 = r, S_2 = r) + P(S_3 = r, S_2 = s) + P(S_3 = r, S_2 = c) = P(S_3 = r | S_2 = r)P(S_2 = r) + P(S_3 = r | S_2 = s)P(S_2 = s) + P(S_3 = r | S_2 = c)P(S_2 = c) \]

Only hard part: figure out \( P(S_2) \)

Main idea: compute distribution \( P(S_1) \), then \( P(S_2) \), then \( P(S_3) \)

Linear in number of periods!
Stationary distributions

• As t goes to infinity, “generally,” the distribution $P(S_t)$ will converge to a stationary distribution

• A distribution given by probabilities $\pi_i$ (where i is a state) is stationary if:
  
  $P(S_t = i) = \pi_i$ means that $P(S_{t+1} = i) = \pi_i$

• Of course,
  
  $P(S_{t+1} = i) = \sum_j P(S_{t+1} = i, S_t = j) = \sum_j P(S_t = j) a_{ji}$

• So, stationary distribution is defined by
  
  $\pi_i = \sum_j \pi_j a_{ji}$
Computing the stationary distribution

\[ \pi_s = 0.6\pi_s + 0.4\pi_c + 0.2\pi_r \]
\[ \pi_c = 0.3\pi_s + 0.3\pi_c + 0.5\pi_r \]
\[ \pi_r = 0.1\pi_s + 0.3\pi_c + 0.3\pi_r \]
Restrictiveness of Markov models

- Are past and future really independent given current state?
- E.g., suppose that when it rains, it rains for at most 2 days

Second-order Markov process
- Workaround: change meaning of “state” to events of last 2 days

Another approach: add more information to the state
- E.g., the full state of the world would include whether the sky is full of water
  - Additional information may not be observable
  - Blowup of number of states…
Hidden Markov models (HMMs)

• Same as Markov model, except we cannot see the state

• Instead, we only see an observation each period, which depends on the current state

\[ S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow \ldots \rightarrow S_t \rightarrow \ldots \]

\[ O_1 \rightarrow O_2 \rightarrow O_3 \rightarrow \ldots \rightarrow O_t \rightarrow \ldots \]

• Still need a transition model: \( P(S_{t+1} = j \mid S_t = i) = a_{ij} \)

• Also need an observation model: \( P(O_t = k \mid S_t = i) = b_{ik} \)
Weather example extended to HMM

- Transition probabilities:

\[
\begin{align*}
    p_{sw} &= 0.3, \\
p_{cw} &= 0.4, \\
p_{rw} &= 0.2, \\
p_{sc} &= 0.3, \\
p_{sr} &= 0.6, \\
p_{rs} &= 0.1, \\
p_{sr} &= 0.5
\end{align*}
\]

- Observation: labmate wet or dry

- \( b_{sw} = 0.1, b_{cw} = 0.3, b_{rw} = 0.8 \)
HMM weather example: a question

• You have been stuck in the lab for three days (!)
• On those days, your labmate was dry, wet, wet, respectively
• What is the probability that it is now raining outside?
• $P(S_2 = r \mid O_0 = d, O_1 = w, O_2 = w)$
• By Bayes’ rule, really want to know $P(S_2, O_0 = d, O_1 = w, O_2 = w)$

\[ b_{sw} = .1 \]
\[ b_{cw} = .3 \]
\[ b_{rw} = .8 \]
Solving the question

- Computationally efficient approach: first compute $P(S_1 = i, O_0 = d, O_1 = w)$ for all states $i$
- General case: solve for $P(S_t, O_0 = o_0, O_1 = o_1, ..., O_t = o_t)$ for $t=1$, then $t=2$, ... This is called monitoring
- $P(S_t, O_0 = o_0, O_1 = o_1, ..., O_t = o_t) = \sum_{s_{t-1}} P(S_{t-1} = s_{t-1}, O_0 = o_0, O_1 = o_1, ..., O_{t-1} = o_{t-1}) P(S_t | S_{t-1} = s_{t-1}) P(O_t = o_t | S_t)$

\[ b_{sw} = 0.1 \]
\[ b_{cw} = 0.3 \]
\[ b_{rw} = 0.8 \]
Predicting further out

- You have been stuck in the lab for three days
- On those days, your labmate was dry, wet, wet, respectively
- What is the probability that two days from now it will be raining outside?
- \( P(S_4 = r \mid O_0 = d, O_1 = w, O_2 = w) \)
Predicting further out, continued…

- Want to know: $P(S_4 = r \mid O_0 = d, O_1 = w, O_2 = w)$
- Already know how to get: $P(S_2 \mid O_0 = d, O_1 = w, O_2 = w)$
- $P(S_3 = r \mid O_0 = d, O_1 = w, O_2 = w) = \sum_{s_2} P(S_3 = r, S_2 = s_2 \mid O_0 = d, O_1 = w, O_2 = w)$
- $\sum_{s_2} P(S_3 = r \mid S_2 = s_2)P(S_2 = s_2 \mid O_0 = d, O_1 = w, O_2 = w)$
- Etc. for $S_4$
- So: monitoring first, then straightforward Markov process updates

\[
\begin{align*}
b_{sw} &= .1 \\
b_{cw} &= .3 \\
b_{rw} &= .8
\end{align*}
\]
• You have been stuck in the lab for **four** days (!)
• On those days, your labmate was dry, wet, wet, dry respectively
• What is the probability that **two days ago** it was raining outside? $P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d)$
  
  – Smoothing or hindsight problem

\[ b_{sw} = .1 \]
\[ b_{cw} = .3 \]
\[ b_{rw} = .8 \]
Want: $P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d)$

“Partial” application of Bayes’ rule:

$$P(S_1 = r \mid O_0 = d, O_1 = w, O_2 = w, O_3 = d) = \frac{P(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)}{P(O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)}$$

So really want to know $P(S_1, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)$
Hindsight problem continued…

- Want to know $P(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w)$
- $P(S_1 = r, O_2 = w, O_3 = d \mid O_0 = d, O_1 = w) = P(S_1 = r \mid O_0 = d, O_1 = w) \cdot P(O_2 = w, O_3 = d \mid S_1 = r)$
- Already know how to compute $P(S_1 = r \mid O_0 = d, O_1 = w)$
- Just need to compute $P(O_2 = w, O_3 = d \mid S_1 = r)$

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$P(S_1 = r | O_0 = d, O_1 = w)$
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$\begin{align*}
&b_{sw} = .1 \\
b_{cw} = .3 \\
b_{rw} = .8
\end{align*}$
Hindsight problem continued…

Just need to compute $P(O_2 = w, O_3 = d | S_1 = r)$

$P(O_2 = w, O_3 = d | S_1 = r) = \sum_{s_2} P(S_2 = s_2, O_2 = w, O_3 = d | S_1 = r) = \sum_{s_2} P(S_2 = s_2 | S_1 = r) P(O_2 = w | S_2 = s_2) P(O_3 = d | S_2 = s_2)$

First two factors directly in the model; last factor is a “smaller” problem of the same kind

Use dynamic programming, backwards from the future
  – Similar to forwards approach from the past
Backwards reasoning in general

• Want to know \( P(O_{k+1} = o_{k+1}, \ldots, O_t = o_t \mid S_k) \)

• First compute

\[
P(O_t = o_t \mid S_{t-1}) = \sum_{s_t} P(S_t = s_t \mid S_{t-1})P(O_t = o_t \mid S_t = s_t)
\]

• Then compute

\[
P(O_t = o_t, O_{t-1} = o_{t-1} \mid S_{t-2}) = \sum_{s_{t-1}} P(S_{t-1} = s_{t-1} \mid S_{t-2})P(O_{t-1} = o_{t-1} \mid S_{t-1} = s_{t-1}) P(O_t = o_t \mid S_{t-1} = s_{t-1})
\]

• Etc.
Variable elimination

- Because all of this is inference in a Bayes net, we can also just do variable elimination

\[ P(S_3 = r, O_1 = d, O_2 = w, O_3 = w) = \]
\[ \sum_{s_2} \sum_{s_1} P(S_1 = s_1)P(O_1 = d | S_1 = s_1)P(S_2 = s_2 | S_1 = s_1)P(O_2 = w | S_2 = s_2)P(S_3 = r | S_2 = s_2)P(O_3 = w | S_3 = r) \]

- It’s a tree, so variable elimination works well
Dynamic Bayes Nets

- So far assumed that each period has one variable for state, one variable for observation
- Often better to divide state and observation up into multiple variables

edges both within a period, and from one period to the next...
Some interesting things we skipped

• Finding the most likely sequence of states, given observations
  – Not necessary equal to the sequence of most likely states! (example?)
  – Viterbi algorithm
    • Key idea: for each period t, for every state, keep track of most likely sequence to that state at that period, given evidence up to that period

• Continuous variables

• Approximate inference methods
  – Particle filtering